Tesler Matrices and Lusztig Data

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Tesler Matrices

Definition

For a $n \times n$ upper-triangular matrix A with non-negative integer entries, we define its k^{th} ***hook sum*** h_k, $1 \leq k \leq n$ as

$$h_k = \sum_{i=k}^{n} a_{ki} - \sum_{i=1}^{k-1} a_{ik}$$

and its ***hook sum vector*** as $\mathbf{h} = (h_1, h_2, ..., h_n)$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{pmatrix}$$

$$h_1 = 1 \quad h_2 = 1 \quad h_3 = 1 \quad h_4 = 1 \quad h_5 = 1$$

Figure: Hook sums of a Tesler matrix

Definition

The set of all upper triangular matrices with hook sum $\mathbf{h}$ is denoted $T(\mathbf{h})$. Its elements are called ***Tesler matrices***.

Example

For example, the matrix we used before is also Tesler matrix from the set $T(1, 1, 1, 1, 1)$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{pmatrix}$$

Kostant pictures

Definition

In $\mathbb{R}^{n+1}$ with its standard basis $\{\mathbf{e}_i\}_{1\le i\le n}$ we define the ***positive roots*** $\alpha_{ij} = \mathbf{e}_i - \mathbf{e}_{j+1}$ for $1 \le i \le j \le n$. In particular, the ***simple roots*** are $\alpha_i \equiv \alpha_{ii} = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $1 \le i \le n$.

Example

In $\mathbb{R}^5$ we have $\alpha_{24} = (0,\, 1,\, 0,\, 0,\, -1)$ and

$$\begin{aligned}\alpha_{24} = \alpha_2 + \alpha_3 + \alpha_4 &= (0,\, 1,\, -1,\, 0,\, 0)\\ &+ (0,\, 0,\, 1,\, -1,\, 0)\\ &+ (0,\, 0,\, 0,\, 1,\, -1)\,.\end{aligned}$$

Definition

We define ***positive root cone*** of $\mathbb{Z}_{\geq 0}$-linear combinations of simple roots as Q_+.

Example

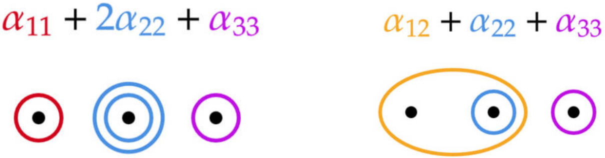
For $\mathbb{R}^2$: $Q_+ = \{a_{11} \cdot \alpha_1 : a_{ij} \in \mathbb{Z}_{\geq 0}\}$.
For $\mathbb{R}^3$: $Q_+ = \{a_{11} \cdot \alpha_1 + a_{12} \cdot \alpha_{12} + a_{22} \cdot \alpha_2 : a_{ij} \in \mathbb{Z}_{\geq 0}\}$.

Definition

A ***Kostant picture*** of weight $v \in Q_+$ is a diagram representing a decomposition of the weight v as a non-negative integer sum of positive roots. We draw n black dots evenly spaced in a line, one for every simple root. The root α_{ij} is represented by uniting dots number i and j with a loop that contains all dots from i to j inclusively.

Example

Let $v = (1, 1, -1, -1)$. Let us emphasize the correspondence between loops and roots. The Kostant picture of the decompositions $\alpha_1 + 2\alpha_2 + \alpha_3$ and $\alpha_{12} + \alpha_2 + \alpha_3$ are provided below.

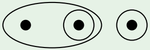


Example

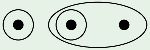
Or here is an example for all possible combinations.

$(\alpha_{11} + 2\alpha_{22} + \alpha_{33})$

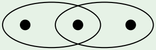
$(\alpha_{12} + \alpha_{22} + \alpha_{33})$



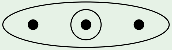
$(\alpha_{11} + \alpha_{22} + \alpha_{23})$



$(\alpha_{12} + \alpha_{23})$



$(\alpha_{13} + \alpha_{22})$



Definition

Given $v \in Q_+$ define its ***Kostant partition function*** $\mathrm{KPF}(v)$ as the number of ways of expressing v as a sum of positive roots.

Example

For example if $v = (1, 1, -1, -1)$ then $\mathrm{KPF}(v) = 5$, since

$$v = \alpha_{11} + 2\alpha_{22} + \alpha_{33} = \alpha_{12} + \alpha_{23} = \alpha_{12} + \alpha_{22} + \alpha_{33} = \alpha_{13} + \alpha_{22} = \alpha_{11} + \alpha_{22} + \alpha_{23}.$$

Lusztig data

Definition

A ***Lusztig data*** is a tuple of $\ell = \binom{n}{2}$ non-negative integers

$$\mathbf{a} = (a_{11}, a_{12}, \dots, a_{1n}, a_{22}, a_{23}, \dots, a_{nn}),$$

having ***weight***

$$\mathrm{wt}(\mathbf{a}) = \sum a_{ij}\alpha_{ij} \in Q_+.$$

Given $v \in Q_+$ we denote the set of all Lusztig data having weight v by $A(v)$.

Lustig data	v decomposition	Lustig data	v decomposition
(1, 0, 0, 2, 0, 1)	$\alpha_{11} + 2\alpha_{22} + \alpha_{33}$	(0, 1, 0, 0, 1, 0)	$\alpha_{12} + \alpha_{23}$
(0, 1, 0, 1, 0, 1)	$\alpha_{12} + \alpha_{22} + \alpha_{33}$	(0, 0, 1, 1, 0, 0)	$\alpha_{13} + \alpha_{22}$
(1, 0, 0, 1, 1, 0)	$\alpha_{11} + \alpha_{22} + \alpha_{23}$		

Table: Lusztig data of weight $v = (1, 1, -1, -1)$

Tesler Poset

Definition (Poset on Tesler matrices, O'Neill)

For a fixed hook sum vector $\mathbf{h}$, let $A \in \mathcal{T}(\mathbf{h})$ cover $B \in \mathcal{T}(\mathbf{h})$ ($A \succeq B$) iff they have the same entries except $a_{ij} = b_{ij} + 1$, $a_{jk} = b_{jk} + 1$, $a_{ik} = b_{ik} - 1$ for a unique triple $i < j < k$ or $a_{ij} = b_{ij} + 1$, $a_{jj} = b_{jj} + 1$, $a_{ii} = b_{ii} - 1$ for a unique pair $i < j$.

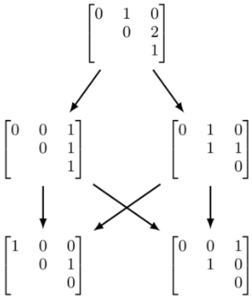


Figure: Poset on $\mathcal{T}(1, 1, -1, -1)$

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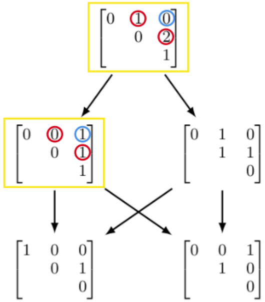

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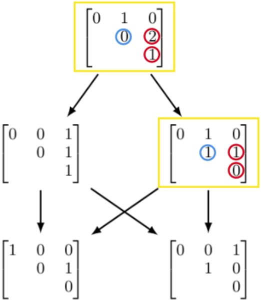

Figure: Poset on $\mathcal{T}(1, 1, -1, -1)$

Lusztig data poset

Definition (Poset via Lusztig data)

The ***double-sided dictionary*** partial order on $A(v)$ is defined by $\mathbf{a} \leq \mathbf{a}'$ if there can be found two integers $l \leq r$ such that

- $a'_l > a_l$,
- $a'_r > a_r$,
- $a'_i = a_i$ for all $i < l$ and $i > r$.

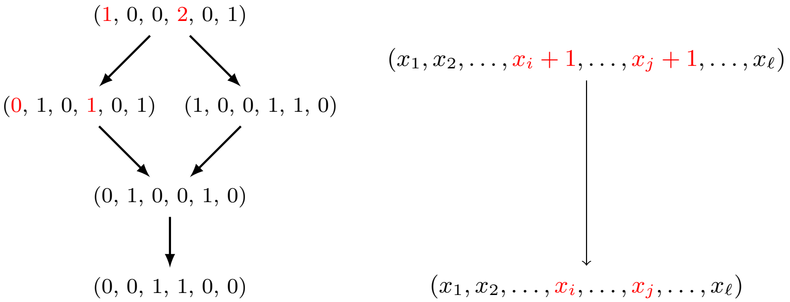

Figure: Poset on $A(1, 1, -1, -1)$

Definition (Poset on Kostant pictures)

We define the additive partial order on the set of all Kostant pictures by the following covering relation: $\alpha_{ij} < \alpha_{ik} + \alpha_{k+1j}$, $i \leq k < j$.

Example

For Kostant pictures $\mathcal{K}(1, 1, -1, -1)$ we can draw the corresponding Hasse diagram.

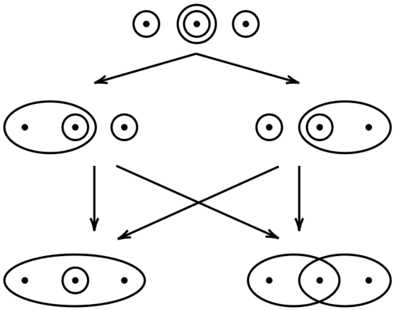

Figure: $\mathcal{K}(1, 1, -1, -1)$

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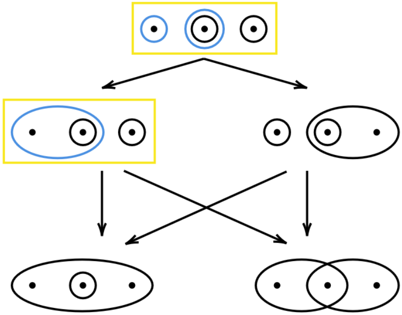


Figure: $\mathcal{K}(1, 1, -1, -1)$

Integral flow

Definition

The ***integral flow graph*** with net flow $\mathbf{h}$ on $n+1$ vertices consist of non-negative flows on the edges and an example of how it looks is shown below. The set of all such graphs is denoted $\mathcal{I}(\mathbf{h})$.

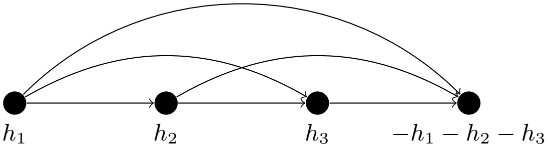


Figure: Example of an integral flow graph

Theorem (Mészáros, Morales, Rhoades)

The sets $\mathcal{T}(\mathbf{h})$ and $\mathcal{I}(\mathbf{h})$ are equivalent.

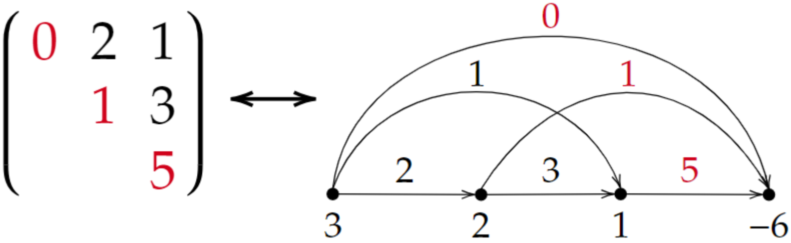


Figure: One element of $\mathcal{T}(3, 2, 1, -6)$ and its integral flow $\mathcal{I}(3, 2, 1, -6)$

- How will the Tesler partial order appear in an integral flow?

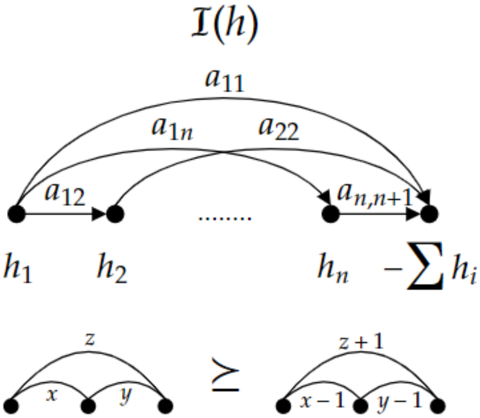
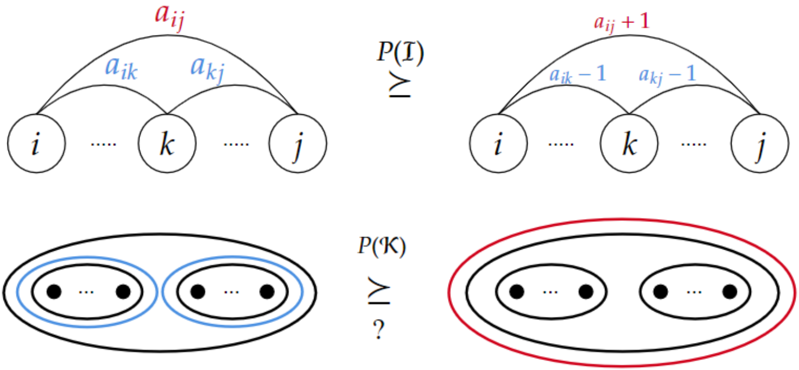


Figure: The corresponding integral flow poset

Theorem (Balashov, Bulavenko, Molybog)

The partial order on $\mathcal{I}(\mathbf{h})$ is isomorphic to the merging order on $\mathcal{K}(\mathbf{h})$.

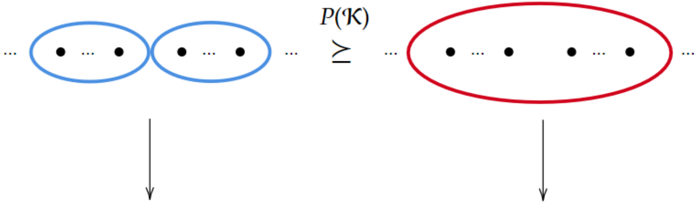


Conjecture (Williams)

The two partial orders on the Kostant pictures (one from definition and the other from Lusztig data) are equivalent.

Theorem (Balashov, Bulavenko, Molybog)

Actually, the poset from $\mathcal{K}(\mathbf{h})$ is a ***weak subposet*** *(the cardinalities are equal, however, the edges are preserved only in one direction) of the poset coming from Lusztig data.*



$$(\dots, \alpha_{i,j}, \dots, \alpha_{i,k}, \dots, \alpha_{j+1,k}, \dots) \geq (\dots, \alpha_{i,j}-1, \dots, \alpha_{i,k}+1, \dots, \alpha_{j+1,k}-1, \dots)$$

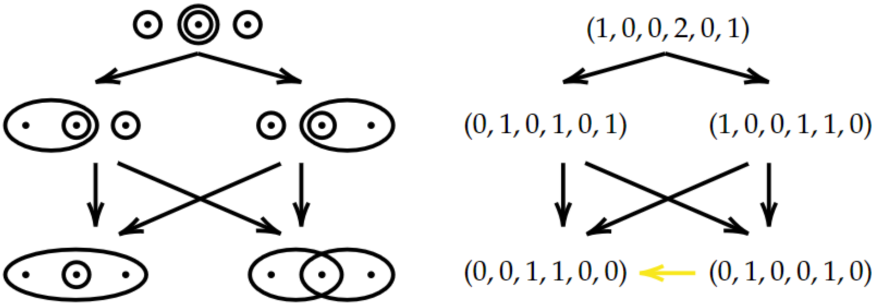


Figure: Counterexample for $\mathbf{h} = (1, 1, -1, -1)$

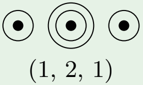
Height diagrams

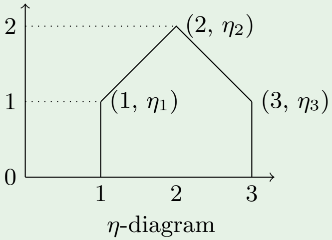
Definition

For a Kostant picture with weight $v = (v_1, v_2, \ldots, v_{n+1})$, we define its ***height*** as $\eta = \left(v_1, v_1 + v_2, \sum_{i=1}^{n} v_i\right)$. Equivalently, η is defined by $v = \sum_{i=1}^{n} \eta_i \alpha_i$.

Example

The height corresponding to $v = (1, 1, -1, -1) = 1\alpha_1 + 2\alpha_2 + 1\alpha_3$ is $\eta = (1, 2, 1)$.



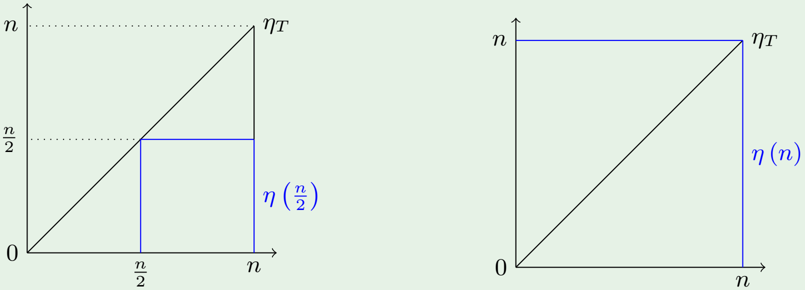


Proposition

$T(\mathbf{1}^n) \approxeq T\left(n, \mathbf{0}^{n-1}\right)$, *where* $f(x) \approxeq g(x)$ *denotes* $\lim_{x\to\infty} \frac{\ln \circ f}{\ln \circ g}(x) \in \mathbb{R}^+$.

Proof.

Note that $\mathbf{h} = \mathbf{1}^n$ corresponds to $\eta_T = (1, 2, ..., n)$, while $\mathbf{h} = \left(n, \mathbf{0}^{n-1}\right)$ corresponds to $\eta(n) = \mathbf{n}^n$.



$$T\left(\frac{n}{2}, \mathbf{0}^{\frac{n}{2}-1}\right) \leq T(\mathbf{1}^n) \leq T\left(n, \mathbf{0}^{n-1}\right)$$

□

Background

Example

For a simple hook sum vector case: $T\left(1,\, \mathbf{0}^{n-1}\right) = 2^{n-1}$.

Result (Zeilberger)

$$T\left(1,\, 2,\, ...,\, n\right) = \prod_{i=1}^{n} C_i \approx e^{n^2},$$

where $C_i = \frac{1}{i+1}\binom{2i}{i}$ *denotes the* i^{th} *Catalan number.*

Conjecture (O'Neill)

Let $\alpha = (1,\, 1,\, ...,\, 1)$ *and* $P\left(\alpha\right)$ *be the Tesler poset with Möbius function* $\mu\left(\cdot\right)$*. Then*

$$\left|\mu\left(\hat{0},\, A\right)\right| \leq n!.$$

If the conjecture holds, $T\left(\mathbf{1}^n\right) \approx e^{n^2}$.

Using height diagrams, it is easy to show $T\left(n^2, \mathbf{0}^{n-1}\right) \approx T(1, 2, ..., n) \approx e^{n^2}$, meaning $T\left(n, \mathbf{0}^{\sqrt{n}-1}\right) \approx e^n$.

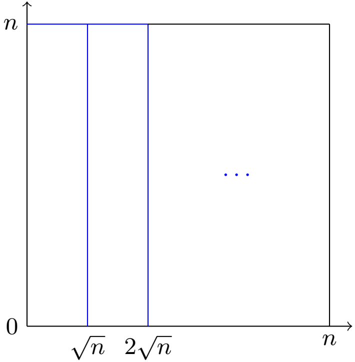


Figure: Decomposition of $T\left(n, \mathbf{0}^{n-1}\right)$

$$e^{n\sqrt{n}} \lessapprox T\left(n, \mathbf{0}^{n-1}\right) \lessapprox n^{n\sqrt{n}}$$

Theorem (Balashov, Bulavenko, Molybog)

$T\left(\mathbf{1}^n\right) \approx T\left(n, \mathbf{0}^{n-1}\right) \approx e^{n\sqrt{n}}$, *disproving O'Neill's conjecture.*

Corollary

$T\left(h(x), \mathbf{0}^{w(x)-1}\right) \approx e^{w(x)\sqrt{h(x)}}$, *where* $\lim_{x\to\infty} h(x) = \infty$ *and* $\lim_{x\to\infty} \frac{h(x)}{w(x)^2} \in \mathbb{R}$.

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Thank you for your attention!